Direct Method for the Solution of Linear Equations

Jun-Feng Yin

Tongji University

yinjf@tongji.edu.cn

http://www.tongji.edu.cn/~yin

Direct Method for the Solution of Linear Equations

$\Rightarrow$ Introduction

- Naive Gaussian Elimination
- Limitations and Operation Counts
- LU factorization
- QR factorization

What Are Linear Equations (LEs)?

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ . & + & . & + & \ldots & + & . & = & . \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

- Dependence on unknowns: powers of degree ≤ 1
- Summation form: $\sum_{j=1}^{n} a_{ij}x_j = b_i$, $1 \leq i \leq m$, i.e., m equations
- Presently: $m = n$, i.e., square systems (later: $m \neq n$)

Q: How to solve for $\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$**? A:** $\ldots$

Gaussian Elimination

- Introduction

⇒ Naive Gaussian Elimination

- Limitations and Operation Counts
- LU factorization
- QR factorization

Overall Algorithm and Definitions

- Currently: direct method only (later: iterative methods)
- General idea:
  - * Generate upper triangular system ("forward elimination")
  - * Easily calculate unknowns in reverse order ("backward substitution")
- "Pivot row" = current one being processed
  "pivot" = diagonal element of pivot row
- Steps applied to **RHS** as well. (**RHS**: Right hand side vector.)

Forward Elimination

- Generate zero columns below diagonal

- Process rows downward

```
for each row i := 1, n − 1 {            // the pivot row
    for each row k := i + 1, n {        ∀ rows below pivot
        multiply pivot row a_ii = a_ki
        subtract pivot row from row k       // now a_ki = 0
    }     // now column below a_ii is zero
}    now // a_ij = 0, ∀ i > j
```

- Obtain triangular system

Let's work an example, ...

Matrix Form of Linear Equations

$$
\begin{array}{rcrcrcrcr}
6x_1 & - & 2x_2 & + & 2x_3 & + & 4x_4 & = & 16 \\
12x_1 & - & 8x_2 & + & 6x_3 & + & 10x_4 & = & 26 \\
3x_1 & - & 13x_2 & + & 9x_3 & + & 3x_4 & = & -19 \\
-6x_1 & + & 4x_2 & + & 1x_3 & - & 18x_4 & = & -34
\end{array}
$$

Matrix form

$$
\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}, \quad b
$$

Compact Form of Linear Equations

$$
\begin{aligned}
6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\
12x_1 - 8x_2 + 6x_3 + 10x_4 &= 26 \\
3x_1 - 13x_2 + 9x_3 + 3x_4 &= -19 \\
-6x_1 + 4x_2 + 1x_3 - 18x_4 &= -34
\end{aligned}
$$

Compact form

$$
\begin{array}{ccccc}
6 & -2 & 2 & 4 & 16 \\
12 & -8 & 6 & 10 & 26 \\
3 & -13 & 9 & 3 & -19 \\
-6 & 4 & 1 & -18 & -34
\end{array}
$$

Proceeding with the forward elimination, ...

Forward Elimination–Example

$$
\begin{matrix} 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{matrix}
\rightarrow
\begin{matrix} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & -18 \end{matrix}
\rightarrow
$$

$$
\begin{matrix} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{matrix}
\rightarrow
\begin{matrix} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{matrix}
$$

Matrix is upper triangular.

Upper Sum

$$\begin{pmatrix} 6 & -2 & 2 & 4 & 16 \\ 0 & -4 & 2 & 2 & -6 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{pmatrix}$$

- Last equation: $-3x_4 = -3 \Rightarrow x_4 = 1$
- Second to last equation: $2x_3 - 5\underbrace{x_4}_{=1} = 2x_3 - 5 = -9 \Rightarrow x_3 = -2$
- ... second equation ... $x_2 = \cdots$
- $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 & -2 & 1 \end{pmatrix}^T$

For small problems, check solution in original system.

Linear Systems

- Introduction
- Naive Gaussian Elimination

$\Rightarrow$ Limitations and Operation Counts

- LU factorization
- QR factorization

Zero Pivots

- Clearly, zero pivots prevent forward elimination
- **Attention:** zero pivots can appear along the way
- Later: Where guaranteed no zero pivots?
- All pivots $\neq 0 \Rightarrow^{?}$ we are safe

Experiment with system with known solution.