CHAPTER V

FUNCTIONS OF TWO OR MORE VARIABLES

77. Although thus far we have considered several variable quantities, yet these were prepared thus, so that all were functions of one [variable] and by a single determination the others likewise could be determined. But now we will consider variable quantities of this kind, which may not depend on each other in turn, thus so that, whatever the determined value attributed to one variable, the rest nevertheless still remain undetermined and variable. Therefore variable quantities of this kind – such as *x, y, z* – will agree by reason of the assigned values, since any assigned values you please may themselves be combined together; moreover, if they may be prepared individually, they will be especially diverse, since, whatever determined value you please may be substituted for one value of *z*, yet the remaining *x* and *y* may extend out just as widely as before. Therefore the distinction is shifted from variable quantities depending on each other to this case with variable quantities not depending on each other, as in the first case, if one value may be determined, likewise the rest will be determined; truly in the later case the determination of one variable will restrain minimally the assigned values of the others.

78. *Therefore a function of two or of several variable quantities x, y, z is some kind of expression composed from these quantities.*

Thus

$$x^3 + xyz + az^2$$

will be a function of the three variable quantities *x, y, z*. Therefore this function, if one variable may be determined, e.g. *z*, that is, in place of this a constant number may be substituted, at this point will remain a variable quantity, evidently a function of *x* and *y*. And if besides *z* , *y* also may be determined, then at this stage it will be a function of *x*. Therefore a function of several variables of this kind will not be given any determined value before the individual variable quantities should be determined. Therefore since one variable quantity shall be able to be determined in an infinite number of ways, a function of two variables, because it is possible to take infinitely many determinations for the determination of the one, generally allows an infinitude of infinite determinations. And in a function of three variables the number of determinations at this stage will be infinitely greater ; and thus it will increase again for more variables.

79. *Functions of this kind of several variables and likewise functions of one variable are divided most conveniently into algebraic and transcending forms.*

Of which the former, in which the account of the composition has been put in place in terms of algebraic operations only; the latter truly, in the formation of which transcending operations also are present. In these kinds may be observed anew, provided the transcending operations either implicate all the variable quantities or some or only one. Thus this expression

$$zz + y\log.z\,,$$

because the logarithm of z itself is present, will be indeed a transcending function of y and of z themselves, truly thus it is required to be thought less transcending, because, if the variable z will be determined, it will become an algebraic function of y. Yet meanwhile it will not be expedient to clarify the treatment by subdivisions of this kind.

80. *Algebraic functions then are subdivided into rationals and irrationals, moreover the rational again into whole and fractional.*

The account of these denominations from the first chapter is now understood amply. Clearly a rational function generally is free from all irrationality, of which a function is said to be affected; and this function will be whole, if it is not beset by fractions, otherwise it will be a fractional function. Thus this will be the general form of an integral function of the two variables y and z :

$$\alpha + \beta y + \gamma z + \delta y^2 + \varepsilon yz + \zeta z^2 + \eta y^3 + \theta y^2 z + \iota yz^2 + \chi z^3 + \text{etc.}$$

Therefore if P and Q may denote whole functions of this kind, either of two or of more variables, $\frac{P}{Q}$ will be the general form of fractional functions.

Finally an irrational function is either explicit or implicit ; the former by a root sign now have been completely resolved, but the latter are shown by an irresolvable equation. Thus V will be an implicit irrational function of y and z, if it were

$$V^5 = \left(ayz + z^3\right)V^2 + \left(y^4 + z^4\right)V + y^5 + 2ayz^3 + z^5.$$

81. *Multiform functions of several variables thence must be viewed equally with these [multiform functions], which depend on a single variable.*

Thus rational functions will be uniform, because with single variable quantities determined they exhibit a single determined value. P, Q, R, S etc. may denote rational functions or uniform functions of the variables x, y, z and V will be a biform function of the same variables, if there were

$$V^2 - PV + Q = 0$$

for whatever values are attributed to the determined quantities *x, y* and *z*, the function *V* always will have not one but a double value determined. In a like manner *V* will be a tri–form function, if there were

$$V^3 - PV^2 + QV - R = 0,$$

and a function quadri–form, if it had the form

$$V^4 - PV^3 + QV^2 - RV + S = 0;$$

and in this manner an account of multiform functions of higher degrees will be provided.

82. Just as, if a function of one variable *z* is put equal to zero, the value of the variable quantity *z* determined follows to be either simple or multiple, thus, if a function of the two variables *y* et *z* is put equal to zero, then either variable is defined by the other and thus a function of this variable emerges, since before they were not mutually dependent. In a similar manner, if a function of three variables *x, y, z* is put in place equal to zero, then one variable is defined by the two remaining variables and a function of these exists. Likewise it comes about, if a function may not be put equal to zero, but to some constant quantity or also equal to other functions; for from any equation, however many variables it involves, always one variable is defined by the remaining and it shall be a function of these ; moreover two diverse equations arising between the same variables define two variables by the others, and thus henceforth.

83. *Moreover the division of functions of two or more variables into homogeneous and heterogeneous forms is especially noteworthy.*

A homogeneous function is one that has the same number of dimensions everywhere ; but a *heterogeneous function* is one in which diverse numbers of dimensions occur. Truly each variable is agreed to constitute a single dimension ; and of each square term produced from two variables, two dimensions ; three dimensions are produced from three variable, either from the same [repeated] or diverse variables, and so on thus ; but constant quantities are not admitted into the numeration of dimensions. Thus in these formulas

$$\alpha y,\ \beta z$$

a single dimension is said to be present; in these

$$\alpha y^2,\ \beta yz,\ \gamma z^2$$

indeed two dimensions are present; in these

$$\alpha y^3,\ \beta y^2 z,\ \gamma yz^2,\ \delta z^3$$

three; in these truly

$$\alpha y^4,\ \beta y^3 z,\ \gamma y^2 z^2,\ \delta yz^3,\ \varepsilon z^4$$

four and thus henceforth.

84. We will apply this distinction first to whole functions and we will consider only two variables to be present, because the account of several variables is the same.

Therefore a whole function will be homogeneous, in the individual terms of which the same number of dimensions arises.

Therefore functions of this kind can be subdivided most conveniently following the number of dimensions, which the variables constitute in these everywhere. Thus

$$\alpha y + \beta z$$

will be the general form of a whole function of one dimension ; truly this expression

$$\alpha y^2 + \beta yz + \gamma z^2$$

will be the general form of a function of two dimensions ; then the general form of a [homogeneous] function of three variables will be

$$\alpha y^3 + \beta y^2 z + \gamma yz^2 + \delta z^3$$

of four dimensions truly this

$$\alpha y^4 + \beta y^3 z + \gamma y^2 z^2 + \delta yz^3 + \varepsilon z^4$$

and thus henceforth. Therefore in analogy a constant quantity α alone will be a quantity of zero dimensions.

85. *Again a fractional function will be homogeneous, if its numerator and denominator should be homogeneous functions.*

Thus this fraction

$$\frac{ayz+bzz}{\alpha y+\beta z}$$

will be a homogeneous function of *y* and *z;* moreover the number of dimensions will be found if the number of the dimensions of the denominator is taken from the number of the dimensions of the numerator, and for that reason the proposed fractional function will be of one dimension. Truly this fractional function

$$\frac{y^5+z^5}{yy+zz}$$

will be a [homogeneous] function of three dimensions. Therefore when the same number of dimensions arises in the numerator and in the denominator, then the fraction will be a function of zero dimensions, as happens in this fraction

$$\frac{y^3+z^3}{yyz}$$

or also in these

$$\frac{y}{z},\ \frac{\alpha zz}{yy},\ \frac{\beta y^3}{z^3}$$

But if therefore the dimensions in the denominator shall be more than in the numerator, the number of dimensions of the fraction will be negative ; thus

$$\frac{y}{zz}$$

will be a function of –1 dimensions,

$$\frac{y+z}{y^4+z^4}$$

will be a function of –3 dimensions,

$$\frac{1}{y^5+ayz^4}$$

will be a function of – 5 dimensions, because in the numerator no dimension is present. Moreover, it is understood spontaneously how several homogeneous functions, in which individual function the same number of dimensions rules, either added or subtracted producing a homogeneous function also of the same number of dimensions. Thus this expression

$$\alpha y+\frac{\beta zz}{y}+\frac{\gamma y^4-\delta z^4}{yyz+yzz}$$

will be a function of one dimension ; but this

$$\alpha+\frac{\beta y}{z}+\frac{\gamma zz}{yy}+\frac{y+zz}{yy-zz}$$

will be a function of zero dimensions.

86. The nature of homogeneous functions also can be extended to irrational functions. For if P were some homogeneous function, for example of dimensions n, then $\sqrt{P}$ will be a function of dimensions $\frac{1}{2}n$, $\sqrt[3]{P}$ will be a function of dimensions $\frac{1}{3}n$ and generally $P^{\frac{\mu}{\nu}}$ will be a function of dimensions $\frac{\mu}{\nu}n$. Thus $\sqrt{(yy+zz)}$ will be a function of one dimension, $\sqrt[3]{(y^9+z^9)}$ will be a function of three dimensions, $(yz+zz)^{\frac{3}{4}}$ will be of $\frac{3}{2}$ dimensions and $\frac{yy+zz}{\sqrt{(y^4+z^4)}}$ will be a function of zero dimensions. Therefore from these with the preceding jointly, this expression is understood

$$\frac{1}{y}+\frac{y\sqrt{(yy+zz)}}{z^3}-\frac{y}{\sqrt[3]{(y^6-z^6)}}+\frac{y\sqrt{z}}{zz\sqrt{y}+\sqrt{(y^5+z^5)}}$$

to be a homogeneous function of dimensions -1.

87. Whether an implicit irrational function shall be homogeneous or not, can be deduced from these easily. Let V be an implicit function of this kind and

$$V^3+PV^2+QV+R=0$$

with P, Q and R present functions of y and z themselves. In the first place therefore it is apparent that V cannot be a homogeneous function, unless P, Q and R shall be homogeneous functions. In addition truly, if we may put V to be a function of n dimensions, V^2 will be a function of $2n$ and V^3 a function of $3n$ dimensions ; therefore since it must have the same number of dimensions everywhere, it requires that P shall be a function of n dimensions, Q a function of $2n$ dimensions and R a function of $3n$ dimensions. Therefore if in turn the letters P, Q, R shall be homogeneous functions respectively of $n, 2n, 3n$ dimensions, hence it may be concluded V to be a function of n dimensions. Thus if there were

$$V^5+\left(y^4+z^4\right)V^3+\alpha y^8V-z^{10}=0,$$

V would be a homogeneous function of two dimensions of y and z themselves.

88. *If V were a homogeneous function of y and z of n dimensions, and in that there is put everywhere* $y = uz$ *, the function V will change into a product of the power* z^n *into a certain function of the variable u.*

For by this substitution $y = uz$ powers of *z* of such a size will be introduced into the individual term, as they were present before of *y* itself. Therefore since in the individual terms the dimensions of *y* and *z* jointly shall equal the number *n*, now the variable *z* alone everywhere will have *n* dimensions and thus the power z^n of this will be present everywhere. Therefore the function *V* becomes divisible by this power, and the quotient will be a variable function involving only *u*.

This will be apparent initially with whole functions. For if there shall be

$$V = \alpha y^2 + \beta y^2 z + \gamma yz^2 + \delta z^3 ,$$

on putting $y = uz$ it becomes

$$V = z^3 \left(\alpha u^3 + \beta u^2 + \gamma u + \delta \right) .$$

Then truly likewise it is clear with fractional functions. For let

$$V = \frac{\alpha y + \beta z}{yy + zz} ,$$

evidently a function of -1 dimensions; with the substitution $y = uz$ made it becomes

$$V = z^{-1} \cdot \frac{\alpha u + \beta}{uu + 1}$$

Nor also are irrational functions thus being excepted. Indeed if there shall be

$$v = \frac{y + \sqrt{(yy + zz)}}{z\sqrt{\left(y^3 + z^3\right)}} ,$$

which is a function of $-\frac{3}{2}$ dimensions, on putting $y = uz$ it will produce

$$v = z^{-\frac{3}{2}} \cdot \frac{u + \sqrt{(uu + 1)}}{\sqrt{\left(u^3 + 1\right)}} .$$

And thus in this manner homogeneous functions of only two variables will be reduced to functions of one variable ; nor indeed the power of *z*, which is a factor, corrupts that function of *u*.