Consensus Tracking Under Directed Interaction Topologies: Algorithms and Experiments

Wei Ren, *Member, IEEE*

—Consensus tracking problems with, respectively, bounded control effort and directed switching interaction topologies are considered when a time-varying consensus reference state is available to only a subgroup of a team and the team members have only local interaction. A consensus tracking algorithm explicitly accounting for bounded control effort is proposed and analyzed under a directed fixed interaction topology. Furthermore, convergence analysis for a consensus tracking algorithm is provided when the time-varying consensus reference state is available to a dynamically changing subgroup of the team under directed switching inter-vehicle interaction topologies. Experimental results of a formation control application are demonstrated on a multi-robot platform to validate one of the proposed consensus tracking algorithms.

***Index Terms*—Consensus, cooperative control, formation control, multi-vehicle systems.**

I. Introduction

COOPERATIVE control of multi-vehicle systems has received significant attention in recent years due to its numerous potential applications in space-based interferometers, combat, surveillance, and reconnaissance systems, hazardous material handling, and distributed reconfigurable sensor networks. Cooperative control often requires that individual vehicles share a consistent view of the objectives and the world. For example, a cooperative rendezvous task requires that each vehicle know the rendezvous point. Consensus algorithms guarantee that vehicles sharing information have a consistent view of information that is critical to the coordination task. The instantaneous value of that information is the information state. To achieve consensus, a vehicle updates the value of its information state based on the information states of its local neighbors. The goal is to design a distributed update law so that the information states of all of the vehicles in the network converge to a common value (see [1], [2], and references therein). Consensus-type techniques have been used to solve formation control problems in [3]–[8] and flocking problems in [9], [10], to name a few.

Current research in consensus algorithms primarily assumes that the consensus equilibrium is a weighted average or a weighted power mean of the initial information states and therefore constant. This assumption might not be appropriate when each vehicle's information state evolves over time, as occurs in formation maneuvering problems. In addition, many consensus algorithms ensure only that the information states converge to a common value but does not allow specification of a particular value. While this paradigm is useful for applications such as cooperative rendezvous where there is not a single correct value, there are many applications where there is a desired, or reference, information state. In this case, the convergence issues include both convergence to a common value, as well as convergence of the common state to its reference value. Existing consensus-based formation control approaches are often limited to formation stabilization problems, where the vehicles are regulated to constant locations. To guarantee the formation to track a time-varying trajectory, current approaches often rely on the assumption that all vehicles know the time-varying group reference trajectory or velocity. Existing consensus-based flocking algorithms either lack a group reference or require each vehicle to know the group reference. It is hence interesting to study consensus tracking problems when a time-varying reference is available to only a subgroup of a team and the team members have only local interaction.

Consensus with a constant leader state under undirected switching inter-vehicle interaction topologies is addressed in the leader following case of [11]. Consensus with a constant leader state is further considered in [12] and [13] in the context of a directed fixed interaction topology. Dynamic consensus is studied in [14], where an input signal is available to each agent in the team. Estimation algorithms for dynamic average consensus are studied in [15], where a proportional algorithm and a proportional-integral algorithm are analyzed. A consensus problem with a time-varying leader state is solved in [16] under a variable undirected interaction topology, where it is assumed that the leader's acceleration input is available to each follower in the team. In [17], consensus tracking algorithms are proposed and analyzed under a directed fixed interaction topology, where a time-varying *consensus reference state* is available to only a subset of the team members, called the *group leaders*. The consensus tracking algorithms are also applied in [8] to a formation control problem under a directed fixed interaction topology.

While [8] and [17] have addressed the consensus tracking problem with a time-varying consensus reference state, the algorithms in [8] and [17] do not explicitly account for bounded control effort. Furthermore, the convergence analysis in [17] is restricted to the case of a directed fixed interaction topology. In practice, both the group leaders and the directed inter-vehicle interaction topologies may be dynamically switching.

In this brief, we first propose a consensus tracking algorithm to account for bounded control effort and provide convergence analysis in the case of fixed group leaders and a directed fixed

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The author is with the Electrical and Computer Engineering Department, Utah State University, Logan, UT 84322 USA (e-mail: wren@engineering.usu.edu).

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inter-vehicle interaction topology. We then provide convergence analysis for a consensus tracking algorithm in the case of dynamically changing group leaders and directed switching inter-vehicle interaction topologies. Finally, we experimentally implement and validate a consensus tracking algorithm for a formation control problem on our multi-robot platform in the case of dynamically changing group leaders and directed switching inter-vehicle interaction topologies. These results extend the consensus tracking results in [8] and [17]. All of the results in this brief are based on directed interaction topologies. It is worthwhile to mention that an undirected interaction topology is a special case of a directed interaction topology. A preliminary version of this brief was presented at the 2008 American Control Conference [27].

II. Graph Theory Notions

A weighted graph consists of a node set $\mathcal{V} = \{1, \ldots, p\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A_p = [a_{ij}] \in \mathbb{R}^{p\times p}$. An edge (i,j) in a weighted directed graph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. In contrast, the pairs of nodes in a weighted undirected graph are unordered, where an edge (i,j) denotes that vehicles i and j can obtain information from one another. The weighted adjacency matrix A_p of a weighted directed graph is defined such that a_{ij} is a positive weight if $(j,i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j,i) \notin \mathcal{E}$. The weighted adjacency matrix A_p of a weighted undirected graph is defined analogously except that $a_{ij} = a_{ji}, \forall i \neq j$, since $(j,i) \in \mathcal{E}$ implies $(i,j) \in \mathcal{E}$. If the weights are not relevant, then a_{ij} is set equal to 1 for all $(j,i) \in \mathcal{E}$. In this brief, self edges are not allowed, i.e., $a_{ii} = 0$.

For an edge (i,j) in a directed graph, i is the parent node and j is the child node. A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, which has no parent, and the root has a directed path to every other node. A directed spanning tree of a directed graph is a directed tree that contains all nodes of the directed graph. A directed graph has or contains a directed spanning tree if there exists a directed spanning tree as a subset of the directed graph. That is, there exists at least one node having a directed path to all of the other nodes.

Let the (*nonsymmetric*) Laplacian matrix $L_p = [\ell_{ij}] \in \mathbb{R}^{p\times p}$ be defined as [18] $\ell_{ii} = \sum_{j=1, j\neq i}^{p} a_{ij}$ and $\ell_{ij} = -a_{ij}, i \neq j$. The matrix L_p satisfies the conditions

$$\ell_{ij} \leq 0, \quad i \neq j, \quad \sum_{j=1}^{p} \ell_{ij} = 0, \quad i = 1, \ldots, p. \tag{1}$$

For an undirected graph, L_p is symmetric positive semi-definite. However, L_p for a directed graph is not necessarily symmetric. In both the undirected and directed cases, 0 is an eigenvalue of L_p with the associated eigenvector $\mathbf{1}_p$, where $\mathbf{1}_p$ is a $p \times 1$ column vector of all ones.

Given a matrix $S = [s_{ij}] \in \mathbb{R}^{p\times p}$, the directed graph of S, denoted by $\Gamma(S)$, is the directed graph on p nodes i, $i \in \{1, 2, \ldots, p\}$, such that there is an edge in $\Gamma(S)$ from node j to node i if and only if $s_{ij} \neq 0$ (cf. [19]). Again, we assume that there is no self edge (i,i).

III. Consensus Tracking Algorithms

Suppose that there are n vehicles in a team and the information states of all vehicles satisfy single-integrator kinematics given by

$$\dot{\xi}_i = u_i, \quad i = 1, \ldots, n \tag{2}$$

where $\xi_i \in \mathbb{R}^m$ is the information state of the ith vehicle and $u_i \in \mathbb{R}^m$ is the control input.

The objective of this brief is to design distributed control laws for $u_i, i = 1, \ldots, n$, such that the information states of all vehicles converge to a time-varying consensus reference state with, respectively, bounded control effort and directed switching interaction topologies when the time-varying consensus reference state is available to only a subgroup of the team and the team members have only local interaction.

Suppose that the consensus reference state, denoted by ξ^r, satisfies

$$\dot{\xi}^r = f(t, \xi^r) \tag{3}$$

where $f(\cdot,\cdot)$ is piecewise continuous in t and locally Lipschitz in ξ^r.

We introduce a virtual vehicle with the states ξ^r and $\dot{\xi}^r$, named vehicle $n+1$ without loss of generality. Let $A_{n+1} = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$ be the weighted adjacency matrix for the $n+1$ vehicles, denoting information flow among vehicles and whether a vehicle has access to the consensus reference state. In particular, $a_{ij} > 0, i, j = 1, \ldots, n$, if vehicle i receives information from vehicle j, $a_{i(n+1)} > 0$ if ξ^r and $\dot{\xi}^r$ are available to vehicle i, and $a_{(n+1)j} = 0, j = 1, \ldots, n+1$. Note that $a_{ii} = 0, \forall i$.

Before moving on, we need the following lemma.

Lemma 3.1: [20] Suppose that $L_p \in \mathbb{R}^{p\times p}$ satisfies the property (1). Then the following conditions are equivalent: 1) L_p has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_p$ and all of the other eigenvalues have positive real parts; 2) the directed graph of L_p has a directed spanning tree; 3) the rank of L_p is $p-1$.

A. Bounded Control Inputs

We first consider consensus tracking with bounded control inputs. We propose a consensus tracking algorithm that guarantees that the maximum control effort is independent of the initial information states $\xi_i(0)$ as

$$u_i = \frac{1}{\eta_i}\left[\sum_{j=1}^{n} a_{ij}\widehat{\dot{\xi}}_j + a_{i(n+1)}\dot{\xi}^r\right] - \frac{1}{\eta_i}\gamma \tanh\left[\sum_{j=1}^{n} a_{ij}(\xi_i - \xi_j) + a_{i(n+1)}(\xi_i - \xi^r)\right] \tag{4}$$

where $i = 1, \ldots, n$, a_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n+1$, is the (i,j)th entry of the weighted adjacency matrix A_{n+1}, $\eta_i \triangleq \sum_{j=1}^{n+1} a_{ij}$, $\widehat{\dot{\xi}}_j$ is the estimate of $\dot{\xi}_j$, γ is a constant positive scalar, and $\tanh(\cdot)$ is defined componentwise. Note that inter-vehicle interaction topology. We then provide convergence analysis for a consensus tracking algorithm in the case of dynamically changing group leaders and directed switching inter-vehicle interaction topologies. Finally, we experimentally implement and validate a consensus tracking algorithm for a formation control problem on our multi-robot platform in the case of dynamically changing group leaders and directed switching inter-vehicle interaction topologies. These results extend the consensus tracking results in [8] and [17]. All of the results in this brief are based on directed interaction topologies. It is worthwhile to mention that an undirected interaction topology is a special case of a directed interaction topology. A preliminary version of this brief was presented at the 2008 American Control Conference [27].

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