Homework #4: Brief Solutions

1. On 2-dimensional space with real coordinates (x, y), define position eigenstates $|x, y\rangle$ with $\hat{x}|x, y\rangle = x|x, y\rangle$ and $\hat{y}|x, y\rangle = y|x, y\rangle$ and normalization $\langle x, y|x', y'\rangle = \delta(x-x')\delta(y-y')$; and momentum eigenstates $|p_x, p_y\rangle$ with $\hat{p}_x|p_x, p_y\rangle = p_x|p_x, p_y\rangle$ and $\hat{p}_y|p_x, p_y\rangle = p_y|p_x, p_y\rangle$ and normalization $\langle p_x, p_y|p'_x, p'_y\rangle = \delta(p_x - p'_x)\delta(p_y - p'_y)$.

Here $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar$, and other commutators between them are zero.

The rotations around the origin form the $SO(2)$ group. Denote the counter-clockwise rotation of angle θ by $g(\theta)$, which maps the point (x, y) to $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. It is easy to check that $g(\theta)\cdot g(\theta') = g(\theta+\theta' \mod 2\pi)$, so this is an Abelian group.

(a). (2pts) The unitary operator for $g(\theta)$ is $\hat{g}(\theta) = \int dx \int dy\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\langle x, y|$. Compute the matrix element of $\hat{g}(\theta)$ under the momentum eigenbasis, $\langle p'_x, p'_y|\hat{g}(\theta)|p_x, p_y\rangle$.

(b). (2pts) Compute the generator of this group, $\hat{L}_z \equiv \left[i\frac{\partial}{\partial\theta}\hat{g}(\theta) \right]_{\theta=0}$. Represent the result by the $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ operators. [Hint: this is of course related to the angular momentum]

(c). (3pts) Consider the 2D harmonic oscillator, $\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2)$. Here m, ω are positive constants. It can be viewed as the sum of two independent harmonic oscillators, $\hat{H} = (\frac{\hat{p}_x^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}) + (\frac{\hat{p}_y^2}{2m} + \frac{m\omega^2\hat{y}^2}{2})$. The ladder operators for the x- and y-components can be defined as $\hat{b}_x = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p}_x)$ and $\hat{b}_y = \sqrt{\frac{m\omega}{2\hbar}}(\hat{y} + \frac{i}{m\omega}\hat{p}_y)$. They satisfy the commutation relation of boson annihilation operators, $[\hat{b}_x, \hat{b}_x^\dagger] = [\hat{b}_y, \hat{b}_y^\dagger] = 1$, $[\hat{b}_x, \hat{b}_y^\dagger] = [\hat{b}_x, \hat{b}_y] = 0$. Denote the unique normalized ground state of $\hat{H}$ by $|\mathrm{vac}\rangle$, then $\hat{b}_x|\mathrm{vac}\rangle = \hat{b}_y|\mathrm{vac}\rangle = 0$. Write down all eigenvalues and normalized eigenstates of $\hat{H}$.

(d). (3pts) Rewrite the $\hat{L}_z$ in (b) in terms of the ladder operators in (c). Show that $[\hat{H}, \hat{L}_z] = 0$. {Hint: use $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}] + \hat{C}[\hat{A}, \hat{D}]\hat{B}$. }

(e). (3pts) The "raising" operators $\hat{b}_x^\dagger$ and $\hat{b}_y^\dagger$ form basis of a 2-dimensional representation of the $SO(2)$ group. $\hat{g}(\theta)$ transforms them to their linear combinations,

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Here $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = \mathrm{i}\hbar$, and other commutators between them are zero.

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(a). (2pts) The unitary operator for $g(\theta)$ is $\widehat{g(\theta)} = \int \mathrm{d}x \int \mathrm{d}y\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\langle x, y|$. Compute the matrix element of $\widehat{g(\theta)}$ under the momentum eigenbasis, $\langle p'_x, p'_y|\widehat{g(\theta)}|p_x, p_y\rangle$.

(b). (2pts) Compute the generator of this group, $\widehat{L_z} \equiv \left[\mathrm{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0}$. Represent the result by the $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ operators. [Hint: this is of course related to the angular momentum]

(c). (3pts) Consider the 2D harmonic oscillator, $\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2)$. Here m, ω are positive constants. It can be viewed as the sum of two independent harmonic oscillators, $\hat{H} = (\frac{\hat{p}_x^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}) + (\frac{\hat{p}_y^2}{2m} + \frac{m\omega^2\hat{y}^2}{2})$. The ladder operators for the x- and y-components can be defined as $\hat{b}_x = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{\mathrm{i}}{m\omega}\hat{p}_x)$ and $\hat{b}_y = \sqrt{\frac{m\omega}{2\hbar}}(\hat{y} + \frac{\mathrm{i}}{m\omega}\hat{p}_y)$. They satisfy the commutation relation of boson annihilation operators, $[\hat{b}_x, \hat{b}_x^\dagger] = [\hat{b}_y, \hat{b}_y^\dagger] = 1$, $[\hat{b}_x, \hat{b}_y^\dagger] = [\hat{b}_x, \hat{b}_y] = 0$. Denote the unique normalized ground state of $\hat{H}$ by $|\text{vac}\rangle$, then $\hat{b}_x|\text{vac}\rangle = \hat{b}_y|\text{vac}\rangle = 0$. Write down all eigenvalues and normalized eigenstates of $\hat{H}$.

(d). (3pts) Rewrite the $\widehat{L_z}$ in (b) in terms of the ladder operators in (c). Show that $[\hat{H}, \hat{L}_z] = 0$. {Hint: use $[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}] + \hat{C}[\hat{A}, \hat{D}]\hat{B}$. }

(e). (3pts) The "raising" operators $\hat{b}_x^\dagger$ and $\hat{b}_y^\dagger$ form basis of a 2-dimensional representation of the $SO(2)$ group. $g(\theta)$ transforms them to their linear combinations,

$(\hat{g}(\theta)\hat{b}_x^\dagger\hat{g}(\theta)^\dagger, \hat{g}(\theta)\hat{b}_y^\dagger\hat{g}(\theta)^\dagger) = (\hat{b}_x^\dagger, \hat{b}_y^\dagger)\cdot R[g(\theta)]$. Solve this 2×2 representation matrix $R[g(\theta)]$. Check that $R[g(\theta)]\cdot R[g(\theta')] = R[g(\theta+\theta')]$.

(f). (5pts) The Abelian group $SO(2)$ has only 1-dimensional(1-dim'l) irreducible representations(irrep.). We can make two orthonormal linear combinations of $\hat{b}_{x,y}^\dagger$ which are 1-dim'l irrep. of $SO(2)$. Namely there are $(\hat{b}_1^\dagger, \hat{b}_2^\dagger) = (\hat{b}_x^\dagger, \hat{b}_y^\dagger)\cdot U$, where U is a constant 2×2 unitary matrix, so that $(\hat{g}(\theta)\hat{b}_1^\dagger\hat{g}(\theta)^\dagger, \hat{g}(\theta)\hat{b}_2^\dagger\hat{g}(\theta)^\dagger) = (\hat{b}_1^\dagger, \hat{b}_2^\dagger)\cdot \begin{pmatrix} R_1[g(\theta)] & 0 \\ 0 & R_2[g(\theta)]\end{pmatrix}$, and $R_{1,2}[g(\theta)]$ are two 1-dimensional(1×1) representation "matrices". Solve $\hat{b}_{1,2}^\dagger$ in terms of $\hat{b}_{x,y}^\dagger$ (or equivalently solve U), and $R_{1,2}[g(\theta)]$. Rewrite $\hat{H}$ and $\hat{L}_z$ in terms of $\hat{b}_{1,2}^\dagger$ and $\hat{b}_{1,2}$.

(g). (3pts) With previous results, write down the simultaneous eigenstates of $\hat{H}$ and $\hat{L}_z$ $|\hat{H} = E, \hat{L}_z = \ell\rangle$, in terms of $|vac\rangle$ and ladder operators. What are the possible eigenvalues E and ℓ?

(h). (5pts) Define two hermitian operators $\hat{L}_x = \hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1$ and $\hat{L}_y = -i\hat{b}_1^\dagger\hat{b}_2 + i\hat{b}_2^\dagger\hat{b}_1$. Check that $[\hat{H}, \hat{L}_x] = [\hat{H}, \hat{L}_y] = 0$. Compute the commutators $[\hat{L}_x, \hat{L}_y]$, $[\hat{L}_y, \hat{L}_z]$, $[\hat{L}_z, \hat{L}_x]$, represent the results in terms of linear combinations of $\hat{L}_{x,y,z}$. [Side remark: $SO(2)$ has only 1-dim'l irrep., but $\hat{H}$ has degenerate eigenvalues. In fact $\hat{H}$ has a larger non-Abelian symmetry. The $\hat{L}_{x,y,z}$ are generators of this symmetry group and commute with $\hat{H}$.]

Solution:

(a) $\langle x, y|p_x, p_y\rangle = \frac{1}{2\pi\hbar}e^{i(p_x x+p_y y)/\hbar}$.

Then $\hat{g}(\theta)|p_x, p_y\rangle = \int dx \int dy\, \hat{g}(\theta)|x, y\rangle\langle x, y|p_x, p_y\rangle$

$= \int dx \int dy\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle \cdot \frac{1}{2\pi\hbar}e^{i(p_x x+p_y y)/\hbar}$.

Change the dummy variables to $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$, or equivalently $x = x'\cos\theta + y'\sin\theta$ and $y = -x'\sin\theta + y'\cos\theta$, the Jacobian of this variable change is unity, $|\frac{\partial(x,y)}{\partial(x',y')}| = |\det \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}| = 1$.

$\hat{g}(\theta)|p_x, p_y\rangle = \int dx' \int dy'\, |x', y'\rangle \cdot \frac{1}{2\pi\hbar}e^{i([(p_x\cos\theta - p_y\sin\theta)x' + (p_y\cos\theta + p_x\sin\theta)y']/\hbar}$

$= |p_x\cos\theta - p_y\sin\theta, p_x\sin\theta + p_y\cos\theta\rangle$ is a momentum eigenstate.

$\langle p'_x, p'_y|\hat{g}(\theta)|p_x, p_y\rangle = \delta(p'_x - (p_x\cos\theta - p_y\sin\theta))\cdot\delta(p'_y - (p_x\sin\theta + p_y\sin\theta))$

$(\widehat{g(\theta)}\hat{b}_x^\dagger\widehat{g(\theta)}^\dagger,\ \widehat{g(\theta)}\hat{b}_y^\dagger\widehat{g(\theta)}^\dagger) = (\hat{b}_x^\dagger,\ \hat{b}_y^\dagger)\cdot R[g(\theta)]$. Solve this 2×2 representation matrix $R[g(\theta)]$. Check that $R[g(\theta)]\cdot R[g(\theta')] = R[g(\theta+\theta')]$.

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$= \int\mathrm{d}x\int\mathrm{d}y\,|x\cos\theta - y\sin\theta, x\sin\theta+y\cos\theta\rangle\cdot\frac{1}{2\pi\hbar}e^{\mathbb{i}(p_x x+p_y y)/\hbar}$.

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$\widehat{g(\theta)}|p_x,p_y\rangle = \int\mathrm{d}x'\int\mathrm{d}y'\,|x',y'\rangle\cdot\frac{1}{2\pi\hbar}e^{\mathbb{i}([(p_x\cos\theta - p_y\sin\theta)x' + (p_y\cos\theta + p_x\sin\theta)y']/\hbar}$

$= |p_x\cos\theta - p_y\sin\theta, p_x\sin\theta + p_y\cos\theta\rangle$ is a momentum eigenstate.

$\langle p_x', p_y'|\widehat{g(\theta)}|p_x,p_y\rangle = \delta(p_x' - (p_x\cos\theta - p_y\sin\theta))\cdot\delta(p_y' - (p_x\sin\theta + p_y\sin\theta))$

$= \delta(p_x - (p^0_x\cos\theta + p^0_y\sin\theta)) \cdot \delta(p_y - (-p^0_x\sin\theta + p^0_y\cos\theta))$.

(b) Method #1:

It'll be most clear to consider the action of $\hat{g}(\theta)$ on a generic state $|\psi\rangle$ with wavefunction $\psi(x,y) = \langle x,y|\psi\rangle$.

$\hat g(\theta)|\psi\rangle = \int dx \int dy\, \hat g(\theta)|x,y\rangle\langle x,y|\psi\rangle = \int dx \int dy\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\, \psi(x,y)$

$= \int dx' \int dy'\, |x', y'\rangle\, \psi(x'\cos\theta + y'\sin\theta, -x'\sin\theta + y'\cos\theta)$.

Therefore the action of $\hat g(\theta)$ on wavefunctions $\psi(x,y)$ is

$\hat g(\theta)\ :\quad \psi(x,y) \mapsto \psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.

So the action of $\hat L_z \equiv \left[i\frac{\partial}{\partial\theta}\hat g(\theta)\right]_{\theta=0}$ is

$\left[i\frac{\partial}{\partial\theta}\hat g(\theta)\right]_{\theta=0}\ :\ \psi(x,y) \mapsto i\frac{\partial}{\partial\theta}\psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)\Big|_{\theta=0} = i[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}]\psi(x,y)$.

Compare this with the actions of $\hat x, \hat y, \hat p_x, \hat p_y$, we have $\hat L_z = \frac{1}{\hbar}[\hat x\hat p_y - \hat y\hat p_x]$.

Method #2:

$\hat g(\theta) = \int dx \int dy\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\langle x,y|$

$= \int dp_x \int dp_y \int dx \int dy\, |p_x,p_y\rangle\langle p_x,p_y|x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle$

$= \int dp_x \int dp_y \int dx \int dy\, |p_x,p_y\rangle \frac{e^{-ip_x(x\cos\theta - y\sin\theta) - ip_y(x\sin\theta + y\cos\theta)}}{2\pi\hbar}\langle x,y|$.

Here we have used $\langle p_x,p_y|x,y\rangle = \frac{e^{(-ip_x x - ip_y y)/\hbar}}{2\pi\hbar}$.

Now only the numerical factor involves θ, take derivative with respect to θ,

$\hat L_z \equiv \left[i\frac{\partial}{\partial\theta}\hat g(\theta)\right]_{\theta=0} = \int dp_x \int dp_y \int dx \int dy\, |p_x,p_y\rangle\, \frac{1}{\hbar}(-p_x y + p_y x)\cdot\frac{e^{(-ip_x x - ip_y y)/\hbar}}{2\pi\hbar}\langle x,y|$

$= \int dp_x \int dp_y \int dx \int dy\, |p_x,p_y\rangle\, \frac{1}{\hbar}(-p_x y + p_y x)\langle p_x,p_y|x,y\rangle\langle x,y|$.

Compare this with the diagonal form of $\hat p_{x,y}$ and $\hat x, \hat y$ operators,

$\hat p_x = \int dp_x \int dp_y\, |p_x,p_y\rangle p_x\langle p_x,p_y|$, $\hat p_y = \int dp_x \int dp_y\, |p_x,p_y\rangle p_y\langle p_x,p_y|$, and

$\hat x = \int dx \int dy\, |x,y\rangle x\langle x,y|$, $\hat y = \int dx \int dy\, |x,y\rangle y\langle x,y|$.

We have $\hat L_z = \frac{1}{\hbar}(-\hat p_x\hat y + \hat p_y\hat x)$.

(c) $\hat H = \hbar\omega\cdot(\hat b_x^\dagger\hat b_x + \hat b_y^\dagger\hat b_y + 1) = \hbar\omega\cdot(\hat n_x + \hat n_y + 1)$. Here $\hat n_x = \hat b^\dagger_x\hat b_x$ and $\hat n_y = \hat b^\dagger_y\hat b_y$.

The eigenvalues are $E_{n_x,n_y} = \hbar\omega\cdot(n_x + n_y + 1)$, with normalized eigenstate $|\hat n_x = n_x, \hat n_y = n_y\rangle = \frac{1}{\sqrt{n_x!\,n_y!}}(\hat b_x^\dagger)^{n_x}(\hat b_y^\dagger)^{n_y}|\text{vac}\rangle$. Here n_x, n_y are non-negative integers.

(d)

$= \delta(p_x - (p'_x \cos\theta + p'_y \sin\theta)) \cdot \delta(p_y - (-p'_x \sin\theta + p'_y \cos\theta))$.

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It'll be most clear to consider the action of $\widehat{g(\theta)}$ on a generic state $|\psi\rangle$ with wavefunction $\psi(x,y) = \langle x,y|\psi\rangle$.

$\widehat{g(\theta)}|\psi\rangle = \int \mathrm{d}x \int \mathrm{d}y\, \widehat{g(\theta)}|x,y\rangle\langle x,y|\psi\rangle = \int \mathrm{d}x \int \mathrm{d}y\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\, \psi(x,y)$
$= \int \mathrm{d}x' \int \mathrm{d}y'\, |x',y'\rangle\, \psi(x'\cos\theta + y'\sin\theta, -x'\sin\theta + y'\cos\theta)$.

Therefore the action of $\widehat{g(\theta)}$ on wavefunctions $\psi(x,y)$ is

$\widehat{g(\theta)}: \quad \psi(x,y) \mapsto \psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.

So the action of $\widehat{L}_z \equiv \left[\mathrm{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0}$ is

$\left[\mathrm{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0}: \ \psi(x,y) \mapsto \mathrm{i}\frac{\partial}{\partial\theta}\psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)\big|_{\theta=0} = \mathrm{i}[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}]\psi(x,y)$.

Compare this with the actions of $\hat{x}$,$\hat{y}$,$\hat{p}_x$,$\hat{p}_y$, we have $\widehat{L}_z = \frac{1}{\hbar}[\hat{x}\hat{p}_y - \hat{y}\hat{p}_x]$.

Method #2:

$\widehat{g(\theta)} = \int \mathrm{d}x \int \mathrm{d}y\, |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle\langle x,y|$
$= \int \mathrm{d}p_x \int \mathrm{d}p_y \int \mathrm{d}x \int \mathrm{d}y\, |p_x,p_y\rangle\langle p_x,p_y|x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle$
$= \int \mathrm{d}p_x \int \mathrm{d}p_y \int \mathrm{d}x \int \mathrm{d}y\, |p_x,p_y\rangle \frac{e^{-\mathrm{i}p_x\cdot(x\cos\theta - y\sin\theta) - \mathrm{i}p_y\cdot(x\sin\theta + y\cos\theta)}}{2\pi\hbar}\langle x,y|$.

Here we have used $\langle p_x,p_y|x,y\rangle = \frac{e^{(-\mathrm{i}p_x x - \mathrm{i}p_y y)/\hbar}}{2\pi\hbar}$.

Now only the numerical factor involves θ, take derivative with respect to θ,

$\widehat{L}_z \equiv \left[\mathrm{i}\frac{\partial}{\partial\theta}\widehat{g(\theta)}\right]_{\theta=0} = \int \mathrm{d}p_x \int \mathrm{d}p_y \int \mathrm{d}x \int \mathrm{d}y\, |p_x,p_y\rangle \left(\frac{1}{\hbar}(-p_x y + p_y x)\cdot \frac{e^{(-\mathrm{i}p_x x - \mathrm{i}p_y y)/\hbar}}{2\pi\hbar}\right)\langle x,y|$
$= \int \mathrm{d}p_x \int \mathrm{d}p_y \int \mathrm{d}x \int \mathrm{d}y\, |p_x,p_y\rangle \left(\frac{1}{\hbar}(-p_x y + p_y x)\langle p_x,p_y|x,y\rangle\right)\langle x,y|$.

Compare this with the diagonal form of $\hat{p}_{x,y}$ and $\hat{x}$,$\hat{y}$ operators,

$\hat{p}_x = \int \mathrm{d}p_x \int \mathrm{d}p_y\, |p_x,p_y\rangle p_x\langle p_x,p_y|$, $\hat{p}_y = \int \mathrm{d}p_x \int \mathrm{d}p_y\, |p_x,p_y\rangle p_y\langle p_x,p_y|$, and

$\hat{x} = \int \mathrm{d}x \int \mathrm{d}y\, |x,y\rangle x\langle x,y|$, $\hat{y} = \int \mathrm{d}x \int \mathrm{d}y\, |x,y\rangle y\langle x,y|$.

We have $\widehat{L}_z = \frac{1}{\hbar}(-\hat{p}_x\hat{y} + \hat{p}_y\hat{x})$.

(c) $\hat{H} = \hbar\omega\cdot(\hat{b}_x^\dagger\hat{b}_x + \hat{b}_y^\dagger\hat{b}_y + 1) = \hbar\omega\cdot(\hat{n}_x + \hat{n}_y + 1)$. Here $\hat{n}_x = \hat{b}_x^\dagger\hat{b}_x$ and $\hat{n}_y = \hat{b}_y^\dagger\hat{b}_y$.

The eigenvalues are $E_{n_x,n_y} = \hbar\omega\cdot(n_x + n_y + 1)$, with normalized eigenstate $|\hat{n}_x = n_x, \hat{n}_y = n_y\rangle = \frac{1}{\sqrt{n_x!n_y!}}(\hat{b}_x^\dagger)^{n_x}(\hat{b}_y^\dagger)^{n_y}|\mathrm{vac}\rangle$. Here n_x, n_y are non-negative integers.

(d)

Use $\hat x = \sqrt{\frac{\hbar}{2m\omega}}(\hat b_x + \hat b_x^\dagger)$, $\hat y = \sqrt{\frac{\hbar}{2m\omega}}(\hat b_y + \hat b_y^\dagger)$, $\hat p_x = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat b_x - \hat b_x^\dagger)$, $\hat p_y = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat b_y - \hat b_y^\dagger)$.

$\hat L_z = -i\hat b_x^\dagger \hat b_y + i\hat b_y^\dagger \hat b_x$.

Use $[\hat H, \hat b_x] = -\hbar\omega \hat b_x$, $[\hat H, \hat b_y] = -\hbar\omega \hat b_y$, $[\hat H, \hat b_x^\dagger] = +\hbar\omega\hat b_x^\dagger$, $[\hat H, \hat b_y^\dagger] = +\hbar\omega\hat b_y^\dagger$,

$[\hat H, \hat b_x^\dagger \hat b_y] = [\hat H,\hat b_x^\dagger]\hat b_y + \hat b_x^\dagger[\hat H,\hat b_y] = \hbar\omega \hat b_x^\dagger \hat b_y - \hat b_x^\dagger\cdot\hbar\omega\hat b_y = 0$, and similarly $[\hat H, \hat b_y^\dagger \hat b_x]=0$.

Therefore $[\hat H, \hat L_z]=0$.

In fact, the commutator of creation/annihilation operator bilinears is still a bilinear, $[\sum_{i,j}\hat b_i^\dagger P_{ij}\hat b_j, \sum_{k,\ell}\hat b_k^\dagger Q_{k\ell}\hat b_\ell] = \sum_{i,j,k,\ell}(\hat b_i^\dagger P_{ij}\delta_{jk}Q_{k\ell}\hat b_\ell - \hat b_k^\dagger Q_{k\ell}\delta_{\ell i}P_{ij}\hat b_j) = \sum_{i,\ell}\hat b_i^\dagger([P,Q])_{i\ell}\hat b_\ell$.
Here $[P,Q] \equiv P\cdot Q - Q\cdot P$ is the commutator of the coefficient matrices. Then

$$[\hat H, \hat L_z] = \hbar\omega\cdot[\hat n_x+\hat n_y, -i\hat b_x^\dagger\hat b_y + i\hat b_y^\dagger\hat b_x] = \hbar\omega\cdot(\hat b_x^\dagger, \hat b_y^\dagger)\cdot\left[\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right]\cdot\begin{pmatrix}\hat b_x\\ \hat b_y\end{pmatrix}=0.$$

(e) Method #1:

Consider the action of $(\hat g(\theta)\hat x\hat g(\theta)^\dagger)$ on position basis $|x,y\rangle$, note that $\hat g(\theta)^\dagger = \hat g(-\theta)$, $(\hat g(\theta)\hat x\hat g(\theta)^\dagger)|x,y\rangle = \hat g(\theta)\hat x\,|x\cos\theta+y\sin\theta, -x\sin\theta+y\cos\theta\rangle$
$= \hat g(\theta)(x\cos\theta+y\sin\theta)|x\cos\theta+y\sin\theta,-x\sin\theta+y\cos\theta\rangle = (x\cos\theta+y\sin\theta)|x,y\rangle$. Therefore $(\hat g(\theta)\hat x\hat g(\theta)^\dagger) = \hat x\cos\theta+\hat y\sin\theta$.

Similarly one can show that $(\hat g(\theta)\hat y\hat g(\theta)^\dagger) = -\hat x\sin\theta+\hat y\cos\theta$.

Consider the action of $(\hat g(\theta)\hat p_{x,y}\hat g(\theta)^\dagger)$ on momentum basis $|p_x,p_y\rangle$, and use the result of (a), one can show that $(\hat g(\theta)\hat p_x\hat g(\theta)^\dagger) = \hat p_x\cos\theta+\hat p_y\sin\theta$, $(\hat g(\theta)\hat p_y\hat g(\theta)^\dagger) = -\hat p_x\sin\theta+\hat p_y\cos\theta$.

Then by the definition of $\hat b^\dagger_{x,y}$, we have

$$(\hat g(\theta)\hat b_x^\dagger\hat g(\theta)^\dagger, \hat g(\theta)\hat b_y^\dagger\hat g(\theta)^\dagger) = (\hat b_x^\dagger,\hat b_y^\dagger)\cdot\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$ Namely, $R[g(\theta)] = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$.

Method #2:

Use $\hat g(\theta)=\exp(-i\theta\hat L_z)$, and the Baker-Hausdorff formula.

Use the result of (d), $[\hat L_z, \hat b_x^\dagger] = i\hat b_y^\dagger$, $[\hat L_z,\hat b_y^\dagger] = -i\hat b_x^\dagger$. By mathematical induction,

$$\underbrace{[\hat L_z,[\hat L_z,\ldots[\hat L_z}_{n\text{-fold commutator}},\hat b_x^\dagger]\ldots]] = \begin{cases}\hat b_x^\dagger, & n=2m;\\ i\hat b_y^\dagger, & n=2m+1.\end{cases}$$

$$\underbrace{[\hat L_z,[\hat L_z,\ldots[\hat L_z}_{n\text{-fold commutator}},\hat b_y^\dagger]\ldots]] = \begin{cases}\hat b_y^\dagger, & n=2m;\\ -i\hat b_x^\dagger, & n=2m+1.\end{cases}$$

$\hat g(\theta)\hat b_x^\dagger\hat g(\theta)^\dagger = \sum_{m=0}^\infty \frac{1}{(2m)!}(-i\theta)^{2m}\hat b_x^\dagger + \sum_{m=0}^\infty\frac{1}{(2m+1)!}i(-i\theta)^{2m+1}\hat b_y^\dagger = \cos\theta\cdot\hat b_x^\dagger+\sin\theta\cdot\hat b_y^\dagger$, Use $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b}_x + \hat{b}_x^\dagger)$, $\hat{y} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{b}_y + \hat{b}_y^\dagger)$, $\hat{p}_x = -\mathbb{i}\sqrt{\frac{m\omega\hbar}{2}}(\hat{b}_x - \hat{b}_x^\dagger)$, $\hat{p}_y = -\mathbb{i}\sqrt{\frac{m\omega\hbar}{2}}(\hat{b}_y - \hat{b}_y^\dagger)$.

$\widehat{L}_z = -\mathbb{i}\hat{b}_x^\dagger\hat{b}_y + \mathbb{i}\hat{b}_y^\dagger\hat{b}_x$.

Use $[\hat{H}, \hat{b}_x] = -\hbar\omega\hat{b}_x$, $[\hat{H}, \hat{b}_y] = -\hbar\omega\hat{b}_y$, $[\hat{H}, \hat{b}_x^\dagger] = +\hbar\omega\hat{b}_x^\dagger$, $[\hat{H}, \hat{b}_y^\dagger] = +\hbar\omega\hat{b}_y^\dagger$,

$[\hat{H}, \hat{b}_x^\dagger\hat{b}_y] = [\hat{H}, \hat{b}_x^\dagger]\hat{b}_y + \hat{b}_x^\dagger[\hat{H}, \hat{b}_y] = \hbar\omega\hat{b}_x^\dagger \cdot \hat{b}_y - \hat{b}_x^\dagger \cdot \hbar\omega\hat{b}_y = 0$, and similarly $[\hat{H}, \hat{b}_y^\dagger\hat{b}_x] = 0$.

Therefore $[\hat{H}, \widehat{L}_z] = 0$.

In fact, the commutator of creation/annihilation operator bilinears is still a bilinear, $[\sum_{i,j}\hat{b}_i^\dagger P_{ij}\hat{b}_j, \sum_{k,\ell}\hat{b}_k^\dagger Q_{k\ell}\hat{b}_\ell] = \sum_{i,j,k,\ell}(\hat{b}_i^\dagger P_{ij}\delta_{jk}Q_{k\ell}\hat{b}_\ell - \hat{b}_k^\dagger Q_{k\ell}\delta_{i\ell}P_{ij}\hat{b}_j) = \sum_{i,\ell}\hat{b}_i^\dagger([P,Q])_{i\ell}\hat{b}_\ell$. Here $[P,Q] \equiv P \cdot Q - Q \cdot P$ is the commutator of the coefficient matrices. Then

$[\hat{H}, \widehat{L}_z] = \hbar\omega \cdot [\hat{n}_x + \hat{n}_y, -\mathbb{i}\hat{b}_x^\dagger\hat{b}_y + \mathbb{i}\hat{b}_y^\dagger\hat{b}_x] = \hbar\omega \cdot (\hat{b}_x^\dagger, \hat{b}_y^\dagger) \cdot [\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix}] \cdot \begin{pmatrix} \hat{b}_x \\ \hat{b}_y \end{pmatrix} = 0.$

(e) Method #1:

Consider the action of $(\widehat{g(\theta)}\hat{x}\widehat{g(\theta)}^\dagger)$ on position basis $|x,y\rangle$, note that $\widehat{g(\theta)}^\dagger = \widehat{g(-\theta)}$,

$(\widehat{g(\theta)}\hat{x}\widehat{g(\theta)}^\dagger)|x,y\rangle = \widehat{g(\theta)}\hat{x}|x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta\rangle$

$= \widehat{g(\theta)}(x\cos\theta + y\sin\theta)|x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta\rangle = (x\cos\theta + y\sin\theta)|x,y\rangle$. Therefore $(\widehat{g(\theta)}\hat{x}\widehat{g(\theta)}^\dagger) = \hat{x}\cos\theta + \hat{y}\sin\theta$.

Similarly one can show that $(\widehat{g(\theta)}\hat{y}\widehat{g(\theta)}^\dagger) = -\hat{x}\sin\theta + \hat{y}\cos\theta$.

Consider the action of $(\widehat{g(\theta)}\hat{p}_{x,y}\widehat{g(\theta)}^\dagger)$ on momentum basis $|p_x, p_y\rangle$, and use the result of (a), one can show that $(\widehat{g(\theta)}\hat{p}_x\widehat{g(\theta)}^\dagger) = \hat{p}_x\cos\theta + \hat{p}_y\sin\theta$, $(\widehat{g(\theta)}\hat{p}_y\widehat{g(\theta)}^\dagger) = -\hat{p}_x\sin\theta + \hat{p}_y\cos\theta$.

Then by the definition of $\hat{b}_{x,y}^\dagger$, we have

$(\widehat{g(\theta)}\hat{b}_x^\dagger\widehat{g(\theta)}^\dagger,\ \widehat{g(\theta)}\hat{b}_y^\dagger\widehat{g(\theta)}^\dagger) = (\hat{b}_x^\dagger,\ \hat{b}_y^\dagger) \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Namely, $R[g(\theta)] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Method #2:

Use $\widehat{g(\theta)} = \exp(-\mathbb{i}\theta\widehat{L}_z)$, and the Baker-Hausdorff formula.

Use the result of (d), $[\widehat{L}_z, \hat{b}_x^\dagger] = \mathbb{i}\hat{b}_y^\dagger$, $[\widehat{L}_z, \hat{b}_y^\dagger] = -\mathbb{i}\hat{b}_x^\dagger$. By mathematical induction,

$$\underbrace{[\widehat{L}_z, [\widehat{L}_z, \ldots [\widehat{L}_z, \hat{b}_x^\dagger] \ldots]]}_{n\text{-fold commutator}} = \begin{cases} \hat{b}_x^\dagger, & n = 2m; \\ \mathbb{i}\hat{b}_y^\dagger, & n = 2m+1. \end{cases}$$

$$\underbrace{[\widehat{L}_z, [\widehat{L}_z, \ldots [\widehat{L}_z, \hat{b}_y^\dagger] \ldots]]}_{n\text{-fold commutator}} = \begin{cases} \hat{b}_y^\dagger, & n = 2m; \\ -\mathbb{i}\hat{b}_x^\dagger, & n = 2m+1. \end{cases}$$

$\widehat{g(\theta)}\hat{b}_x^\dagger\widehat{g(\theta)}^\dagger = \sum_{m=0}^{\infty}\frac{1}{(2m)!}(-\mathbb{i}\theta)^{2m}\hat{b}_x^\dagger + \sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\mathbb{i}(-\mathbb{i}\theta)^{2m+1}\hat{b}_y^\dagger = \cos\theta \cdot \hat{b}_x^\dagger + \sin\theta \cdot \hat{b}_y^\dagger$, and $\hat{g}(\theta)\hat{b}_y^\dagger\hat{g}(\theta)^\dagger = \sum_{m=0}^{\infty}\frac{1}{(2m)!}(-\mathrm{i}\theta)^{2m}\hat{b}_y^\dagger + \sum_{m=0}^{\infty}\frac{1}{(2m+1)!}(-\mathrm{i})(-\mathrm{i}\theta)^{2m+1}\hat{b}_x^\dagger = \cos\theta\cdot\hat{b}_y^\dagger - \sin\theta\cdot\hat{b}_x^\dagger$.

$R[g(\theta)]\cdot R[g(\theta')] = R[g(\theta+\theta')]$ can be easily checked using the trigonometric identities, $\cos\theta\cos\theta' - \sin\theta\sin\theta' = \cos(\theta+\theta')$, $\cos\theta\sin\theta' + \sin\theta\cos\theta' = \sin(\theta+\theta')$.

(f) Method #1: direct diagonalization of $R[g(\theta)]$,

$R[g(\theta)]$ has eigenvalues $e^{-\mathrm{i}\theta}$ and $e^{\mathrm{i}\theta}$, with right-eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}$, and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}$, respectively.

$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \mathrm{i} & -\mathrm{i}\end{pmatrix}$, namely $\hat{b}_1^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_x^\dagger + \mathrm{i}\hat{b}_y^\dagger)$, $\hat{b}_2^\dagger = \frac{1}{\sqrt{2}}(\hat{b}_x^\dagger - \mathrm{i}\hat{b}_y^\dagger)$.

And $R_1[g(\theta)] = e^{-\mathrm{i}\theta}$, and $R_2[g(\theta)] = e^{\mathrm{i}\theta}$.

And $\hat{H} = \hbar\omega\cdot(\hat{b}_1^\dagger\hat{b}_1 + \hat{b}_2^\dagger\hat{b}_2 + 1)$, $\hat{L}_z = \hat{b}_1^\dagger\hat{b}_1 - \hat{b}_2^\dagger\hat{b}_2$

Method #2: use "projection operator",

The $SO(2)$ group has 1-dimensional irreducible representations(irrep), $R'_n[g(\theta)] = e^{-\mathrm{i}n\theta}$, labeled by integer n. Try to "project" the basis $\hat{b}_{x,y}^\dagger$ onto this irrep,

$$\sum_{g\in SO(2)}(R'_n[g])^*\cdot\hat{b}\hat{b}_x^\dagger\hat{b}^\dagger = \int_0^{2\pi}d\theta\, e^{\mathrm{i}n\theta}\cdot(\hat{b}_x^\dagger\cos\theta + \hat{b}_y^\dagger\sin\theta) = \begin{cases}\pi\cdot(\hat{b}_x^\dagger + \mathrm{i}\hat{b}_y^\dagger), & n=1;\\ \pi\cdot(\hat{b}_x^\dagger - \mathrm{i}\hat{b}_y^\dagger), & n=-1;\\ 0 & \text{otherwise.}\end{cases}$$

$$\sum_{g\in SO(2)}(R'_n[g])^*\cdot\hat{b}\hat{b}_y^\dagger\hat{b}^\dagger = \int_0^{2\pi}d\theta\, e^{\mathrm{i}n\theta}\cdot(\hat{b}_y^\dagger\cos\theta - \hat{b}_x^\dagger\sin\theta) = \begin{cases}\pi\cdot(\hat{b}_y^\dagger - \mathrm{i}\hat{b}_x^\dagger), & n=1;\\ \pi\cdot(\hat{b}_y^\dagger + \mathrm{i}\hat{b}_x^\dagger), & n=-1;\\ 0, & \text{otherwise.}\end{cases}$$

Then one can see the basis for $n=1$ irrep is proportional to $(\hat{b}_x^\dagger + \mathrm{i}\hat{b}_y^\dagger)$, for $n=-1$ irrep is proportional to $(\hat{b}_x^\dagger - \mathrm{i}\hat{b}_y^\dagger)$. And this 2-dimensional representation does not contain other irreps.

You may have switched the definitions of $\hat{b}_1$ and $\hat{b}_2$, then just exchange the subscripts $_1$ and $_2$ in the above results.

(g) Define $\hat{n}_1 = \hat{b}_1^\dagger\hat{b}_1$, $\hat{n}_2 = \hat{b}_2^\dagger\hat{b}_2$. Then $|\hat{n}_1 = n_1, \hat{n}_2 = n_2\rangle \equiv \frac{1}{\sqrt{n_1!n_2!}}(\hat{b}_1^\dagger)^{n_1}(\hat{b}_2^\dagger)^{n_2}|\text{vac}\rangle$ is the simultaneous eigenstate of $\hat{H}$ and $\hat{L}_z$, with eigenvalues $E = \hbar\omega\cdot(n_1+n_2+1)$, and

and $\widehat{g(\theta)}\hat{b}_y^\dagger\widehat{g(\theta)}^\dagger = \sum_{m=0}^\infty \frac{1}{(2m)!}(-\mathbb{i}\theta)^{2m}\hat{b}_y^\dagger + \sum_{m=0}^\infty \frac{1}{(2m+1)!}(-\mathbb{i})(-\mathbb{i}\theta)^{2m+1}\hat{b}_x^\dagger = \cos\theta\cdot\hat{b}_y^\dagger - \sin\theta\cdot\hat{b}_x^\dagger$.

$R[g(\theta)]\cdot R[g(\theta')] = R[g(\theta+\theta')]$ can be easily checked using the trigonometric identities, $\cos\theta\cos\theta' - \sin\theta\sin\theta' = \cos(\theta+\theta')$, $\cos\theta\sin\theta' + \sin\theta\cos\theta' = \sin(\theta+\theta')$.

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And $R_1[g(\theta)] = e^{-\mathbb{i}\theta}$, and $R_2[g(\theta)] = e^{\mathbb{i}\theta}$.

And $\hat{H} = \hbar\omega\cdot(\hat{b}_1^\dagger\hat{b}_1 + \hat{b}_2^\dagger\hat{b}_2 + 1)$, $\widehat{L}_z = \hat{b}_1^\dagger\hat{b}_1 - \hat{b}_2^\dagger\hat{b}_2$.

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$$\sum_{g\in SO(2)}(R'_n[g])^*\cdot\hat{g}\hat{b}_y^\dagger\hat{g}^\dagger = \int_0^{2\pi}\mathrm{d}\theta\, e^{\mathbb{i}n\theta}\cdot(\hat{b}_y^\dagger\cos\theta - \hat{b}_x^\dagger\sin\theta) = \begin{cases}\pi\cdot(\hat{b}_y^\dagger - \mathbb{i}\hat{b}_x^\dagger), & n=1;\\ \pi\cdot(\hat{b}_y^\dagger + \mathbb{i}\hat{b}_x^\dagger), & n=-1;\\ 0, & \text{otherwise.}\end{cases}$$

Then one can see the basis for $n=1$ irrep is proportional to $(\hat{b}_x^\dagger + \mathbb{i}\hat{b}_y^\dagger)$, for $n=-1$ irrep is proportional to $(\hat{b}_x^\dagger - \mathbb{i}\hat{b}_y^\dagger)$. And this 2-dimensional representation does not contain other irreps.

You may have switched the definitions of $\hat{b}_1$ and $\hat{b}_2$, then just exchange the subscripts $_1$ and $_2$ in the above results.

(g) Define $\hat{n}_1 = \hat{b}_1^\dagger\hat{b}_1$, $\hat{n}_2 = \hat{b}_2^\dagger\hat{b}_2$. Then $|\hat{n}_1 = n_1, \hat{n}_2 = n_2\rangle \equiv \frac{1}{\sqrt{n_1!n_2!}}(\hat{b}_1^\dagger)^{n_1}(\hat{b}_2^\dagger)^{n_2}|\text{vac}\rangle$ is the simultaneous eigenstate of $\hat{H}$ and $\widehat{L}_z$, with eigenvalues $E = \hbar\omega\cdot(n_1+n_2+1)$, and

$\ell = (n_1 - n_2)$. Here $n_{1,2}$ are non-negative integers.

(h) Use the commutator formula for bilinear operators in (d), these three operators satisfy the commutation relation of Pauli matrices.

$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z$, $[\hat{\sigma}_y, \hat{\sigma}_z] = 2i\hat{\sigma}_x$, $[\hat{\sigma}_z, \hat{\sigma}_x] = 2i\hat{\sigma}_y$.

If you have switched the definitions of $\hat{b}_1$ and $\hat{b}_2$ in (f), this would be $[\hat{\sigma}_x, \hat{\sigma}_y] = -2i\hat{\sigma}_z$, $[\hat{\sigma}_y, \hat{\sigma}_z] = -2i\hat{\sigma}_x$, $[\hat{\sigma}_z, \hat{\sigma}_x] = -2i\hat{\sigma}_y$.

2. Considered $\hat{H} = (\hat{f}_1^\dagger \hat{f}_2 + \hat{f}_2^\dagger \hat{f}_3 + \hat{f}_3^\dagger \hat{f}_4 + \hat{f}_4^\dagger \hat{f}_1 + \text{h.c.})$.
Here $\hat{f}_i(\hat{f}_i^\dagger)$ are annihilation(creation) operators for 4 fermion modes, satisfying $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{ij}$ and $\{\hat{f}_i, \hat{f}_j\} = 0$, and h.c. means hermitian conjugate of the previous 4 terms.

The model conserves total particle number $\hat{n} = \sum_{i=1}^4 \hat{f}_i^\dagger \hat{f}_i$, namely $[\hat{H}, \hat{n}] = 0$.

$\hat{H}$ also has the D_4 point group symmetry, generated by

"4-fold rotation" $C_4 : \hat{f}_1 \to \hat{f}_2 \to \hat{f}_3 \to \hat{f}_4 \to \hat{f}_1$, (this means $\hat{C}_4 \hat{f}_1 \hat{C}_4^\dagger = \hat{f}_2$, etc.), and

"principal axis reflection" $\sigma_s : \hat{f}_1 \to \hat{f}_1, \hat{f}_2 \leftrightarrow \hat{f}_4, \hat{f}_3 \to \hat{f}_3$.

This group has 8 elements, and 5 conjugacy classes: $\{\mathbb{1}\}, \{C_4, C_4^3\}, \{C_4^2\}, \{\sigma_s, C_4^2\sigma_s\}, \{\sigma_d \equiv C_4\sigma_s, C_4^3\sigma_s\}$. The character table for the five irreducible representations, $\Gamma_{1,2,3,4,5}$, is

	$\mathbb{1}$	$2C_4$	C_4^2	$2\sigma_s$	$2\sigma_d$
Γ_1	1	1	1	1	1
Γ_2	1	1	1	−1	−1
Γ_3	1	−1	1	1	−1
Γ_4	1	−1	1	−1	1
Γ_5	2	0	−2	0	0

(a) (5pts) A group element $g \in D_4$ will transform $\hat{f}_i^\dagger$ as $\hat{f}_i^\dagger \mapsto \sum_j \hat{f}_j^\dagger \cdot R[g]_{ji}$, where $R[g]$ is the 4×4 representation matrix. Decompose this into irreducible representations. Namely find $\hat{f}_i'^\dagger = \sum_j \hat{f}_j^\dagger \cdot U_{ji}$, where U_{ji} is a 4×4 unitary matrix, so that $\hat{f}_i'^\dagger$ transform under $g \in D_4$ as $\hat{f}_i'^\dagger \mapsto \sum_j \hat{f}_j'^\dagger \cdot R'[g]_{ji}$ with $R'[g]$ block-diagonalized, and each diagonal block is one of the irreducible representations. Solve the new basis $\hat{f}_i'^\dagger$ in terms of $\hat{f}_i^\dagger$ (or equivalently solve U), and the block-diagonalized representation $R'[g]$ for the generators $g = C_4$ and $g = \sigma_s$. [Hint: use the "projection operator" to find the new basis]

$\ell = (n_1 - n_2)$. Here $n_{1,2}$ are non-negative integers.

(h) Use the commutator formula for bilinear operators in (d), these three operators satisfy the commutation relation of Pauli matrices.

$[\widehat{L}_x, \widehat{L}_y] = 2\mathbb{i}\widehat{L}_z$, $[\widehat{L}_y, \widehat{L}_z] = 2\mathbb{i}\widehat{L}_x$, $[\widehat{L}_z, \widehat{L}_x] = 2\mathbb{i}\widehat{L}_y$.

If you have switched the definitions of $\hat{b}_1$ and $\hat{b}_2$ in (f), this would be $[\widehat{L}_x, \widehat{L}_y] = -2\mathbb{i}\widehat{L}_z$, $[\widehat{L}_y, \widehat{L}_z] = -2\mathbb{i}\widehat{L}_x$, $[\widehat{L}_z, \widehat{L}_x] = -2\mathbb{i}\widehat{L}_y$.

2. Considered $\hat{H} = (\hat{f}_1^\dagger \hat{f}_2 + \hat{f}_2^\dagger \hat{f}_3 + \hat{f}_3^\dagger \hat{f}_4 + \hat{f}_4^\dagger \hat{f}_1 + \text{h.c.})$.

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Γ_3	1	-1	1	1	-1
Γ_4	1	-1	1	-1	1
Γ_5	2	0	-2	0	0

(a) (5pts) A group element $g \in D_4$ will transform $\hat{f}_i^\dagger$ as $\hat{f}_i^\dagger \mapsto \sum_j \hat{f}_j^\dagger \cdot R[g]_{ji}$, where $R[g]$ is the 4×4 representation matrix. Decompose this into irreducible representations. Namely find $\hat{f'}_i^\dagger = \sum_j \hat{f}_j^\dagger \cdot U_{ji}$, where U_{ji} is a 4×4 unitary matrix, so that $\hat{f'}_i^\dagger$ transform under $g \in D_4$ as $\hat{f'}_i^\dagger \mapsto \sum_j \hat{f'}_j^\dagger \cdot R'[g]_{ji}$ with $R'[g]$ block-diagonalized, and each diagonal block is one of the irreducible representations. Solve the new basis $\hat{f'}_i^\dagger$ in terms of $\hat{f}_i^\dagger$ (or equivalently solve U), and the block-diagonalized representation $R'[g]$ for the generators $g = C_4$ and $g = \sigma_s$. [Hint: use the "projection operator" to find the new basis]

(b) (5pts) The Hilbert space with fixed total particle number $\hat{n}$ is a representation space of the D_4 group. Assume that the vacuum state $|\text{vac}\rangle$ is invariant under D_4 group. Then the transformation rules for $\hat{f}_i^\dagger$ completely determine the transformation rules for any states, for example C_4 transforms $\hat{f}_1^\dagger \hat{f}_2^\dagger |\text{vac}\rangle \mapsto \hat{f}_2^\dagger \hat{f}_3^\dagger |\text{vac}\rangle$. Decompose the 6-dimensional 2-particle Hilbert space, with occupation basis $\hat{f}_i^\dagger \hat{f}_j^\dagger |\text{vac}\rangle$ for $i < j$, into irreducible representations of D_4. [Hint: one can first work out the 6×6 representation and then change basis to block-diagonalize it; or use the result of (a) to construct the irreducible representation basis]

(c) (4pts) Rewrite $\hat{H}$ in terms of the $\hat{f}'^\dagger_i$ and $\hat{f}'_i$ solved in (a). Solve all the eigenvalues and eigenstates of $\hat{H}$ in the entire Fock space.

Solution:

(a) Check the updated Lecture Notes #4 for a similar problem.

The basis can be chosen as

irrep.	R^0 basis	$R^0[C_4]$	$R^0[\sigma_s]$
Γ_1	$\hat{f}'^\dagger_1 \equiv \hat{\Gamma}^\dagger_1 = \frac{1}{2}(\hat{f}^\dagger_1 + \hat{f}^\dagger_2 + \hat{f}^\dagger_3 + \hat{f}^\dagger_4)$	1	1
Γ_3	$\hat{f}'^\dagger_2 \equiv \hat{\Gamma}^\dagger_3 = \frac{1}{2}(\hat{f}^\dagger_1 - \hat{f}^\dagger_2 + \hat{f}^\dagger_3 - \hat{f}^\dagger_4)$	-1	1
Γ_5	$(\hat{f}'^\dagger_3 \equiv \hat{\Gamma}^\dagger_{5,x} = \frac{1}{2}(\hat{f}^\dagger_1 - \hat{f}^\dagger_3), \hat{f}'^\dagger_4 \equiv \hat{\Gamma}^\dagger_{5,y} = \frac{1}{2}(\hat{f}^\dagger_2 - \hat{f}^\dagger_4))$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The procedures of using "projection operator" are summarized in the following tables,

(b) (5pts) The Hilbert space with fixed total particle number $\hat{n}$ is a representation space of the D_4 group. Assume that the vacuum state $|\text{vac}\rangle$ is invariant under D_4 group. Then the transformation rules for $\hat{f}_i^\dagger$ completely determine the transformation rules for any states, for example C_4 transforms $\hat{f}_1^\dagger \hat{f}_2^\dagger|\text{vac}\rangle \mapsto \hat{f}_2^\dagger \hat{f}_3^\dagger|\text{vac}\rangle$. Decompose the 6-dimensional 2-particle Hilbert space, with occupation basis $\hat{f}_i^\dagger \hat{f}_j^\dagger|\text{vac}\rangle$ for $i < j$, into irreducible representations of D_4. [Hint: one can first work out the 6×6 representation and then change basis to block-diagonalize it; or use the result of (a) to construct the irreducible representation basis]

(c) (4pts) Rewrite $\hat{H}$ in terms of the $\hat{f'}_i^\dagger$ and $\hat{f'}_i$ solved in (a). Solve all the eigenvalues and eigenstates of $\hat{H}$ in the entire Fock space.

Solution:

(a) Check the updated Lecture Notes #4 for a similar problem.

The basis can be chosen as

irrep. R'	basis	$R'[C_4]$	$R'[\sigma_s]$
Γ_1	$\hat{f'}_1^\dagger \equiv \hat{\Gamma}_1^\dagger = \frac{1}{2}(\hat{f}_1^\dagger + \hat{f}_2^\dagger + \hat{f}_3^\dagger + \hat{f}_4^\dagger)$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
Γ_3	$\hat{f'}_2^\dagger \equiv \hat{\Gamma}_3^\dagger = \frac{1}{2}(\hat{f}_1^\dagger - \hat{f}_2^\dagger + \hat{f}_3^\dagger - \hat{f}_4^\dagger)$	$\begin{pmatrix}-1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$
Γ_5	$(\hat{f'}_3^\dagger \equiv \hat{\Gamma}_{5,x}^\dagger = \frac{1}{2}(\hat{f}_1^\dagger - \hat{f}_3^\dagger), \hat{f'}_4^\dagger \equiv \hat{\Gamma}_{5,y}^\dagger = \frac{1}{2}(\hat{f}_2^\dagger - \hat{f}_4^\dagger))$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

The procedures of using "projection operator" are summarized in the following tables,