- CAPM is built on the Markovitz?s (1959) portfolio optimization theory: Investors hold a well-diversi?ed mean-variance e?cient portfolios that minimize risk (variance) for a desired expected return.
- Sharpe (1964) and Lintner (1965) developed a market equilibrium model:
  - the market portfolio lies on the mean-variance frontier, i.e. e?cient;
  - under two more assumptions: all investors have the same expectations, and they can borrow and lend at a risk-free rate.

- CAPM says that the expected return on asset i is given by

$$E(R_i) = R_f + \beta_i \left(E(R_m) - R_f\right) \qquad (1)$$

  where R_m is the return on the market portfolio, R_f the risk-free rate and

$$\beta_i = \frac{Cov(R_i; R_m)}{Var(R_m)}$$

- Using excess returns, $X_i = R_i - R_f$, we write the pricing relation:

$$E(X_i) = \beta_i E(X_m) \qquad (2)$$

  with

$$\beta_i = \frac{Cov(X_i; X_m)}{Var(X_m)}$$

- If the risk-free rate is nonstochastic, (1) and (2) are equivalent.
- In empirical analysis, (2) is usually used.

- In the absence of the risk-free rate, Black (1972) derived a general version of CAPM:

$$E(R_i) = E(R_0) + {}_i\,(E(R_m) \quad E(R_0)) \tag{3}$$

where R_0 is the return on the zero-beta portfolio that is uncorrelated with the market portfolio ($_0 = 0$).[1]

- Rearranging (3), we have

$$E(R_i) = {}_i + {}_i E(R_m) \tag{4}$$

where

$$_i = (1 \quad _i)\, E(R_0)$$

[1] Unobserved zero-beta portfolio makes the analysis more di?cult.

Early test of the Sharpe-Lintner CAPM focused on 3 implications of (2):

1. The intercept is zero;
2. Beta completely captures the cross-sectional variations of expected return;
3. The market risk premium, $E(X_{mt})$, is positive.

Consider the system model:

$$X_t = \alpha + \beta X_{mt} + \varepsilon_t \tag{5}$$

where X_t is an $N \times 1$ vector of excess returns with

$$E(\varepsilon_t) = 0;\; E\left[\varepsilon_t \varepsilon_t'\right] = \Sigma;\; E\left[\varepsilon_t \varepsilon_{t-s}'\right] = 0 \text{ for } s \neq t$$

$$E(X_t) = \mu;\; E(X_{mt}) = \mu_m;\; Var(X_{mt}) = \sigma_m^2;\; Cov(X_{mt}; \varepsilon_t) = 0$$

ML estimation of (5). (May skip)

Assuming the normality of ε_t, we have:

$$f(X_t|X_{mt}) = (2\pi)^{-N/2}|\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(X_t - \alpha - \beta X_{mt})'\Sigma^{-1}(X_t - \alpha - \beta X_{mt})\right]$$

If ε_t are iid, the joint pdf is:

$$f(X_1, \ldots, X_T | X_{m1}, \ldots, X_{mT}) = \prod_{t=1}^{T} f(X_t|X_{mt})$$

So the log-likelihood function is:

$$\begin{aligned} L = & -\frac{NT}{2}\ln(2\pi) - \frac{T}{2}\ln|\Sigma| \\ & -\frac{1}{2}\sum_{t=1}^{T}(X_t - \alpha - \beta X_{mt})'\Sigma^{-1}(X_t - \alpha - \beta X_{mt}) \end{aligned}$$

Solving the FOCs for and we obtain (the same as OLS):

$$\hat{} = \frac{\sum_{t=1}^{T} X_{mt} - X_m \quad X_t - X}{\sum_{t=1}^{T} X_{mt} \quad X_m{}^2}$$

$$\hat{} = X \quad \hat{}X_m$$

where $X_m = T^{1} \sum_{t=1}^{T} X_{mt}$; and $X = T^{1} \sum_{t=1}^{T} X_t$, and

$$\hat{} = \frac{1}{T} \sum_{t=1}^{T} X_t \quad \hat{} \quad \hat{}X_{mt} \qquad X_t \quad \hat{} \quad \hat{}X_{mt} \quad {}^{0}$$

The asymptotic distributions are:

$$\hat{} \quad N \quad ; \frac{1}{T} \quad 1 + \frac{^2_m}{^2_m}$$

$$\hat{} \quad N \quad ; \frac{1 \quad 1}{T \quad ^2_m}$$

$$T \, \hat{} \quad W \, (T \quad 2; \quad)$$

where W denotes a Wishart distribution.

In the Sharpe-Lintner CAPM, all elements of should be zero, in which case then m is the tangency portfolio.
We now test the null hypothesis:

$$H_0 : \quad = 0 \text{ vs } H_1 : \quad 6= ($$

Wald test statistic is given by

$$J_0 = \hat{}^0[Var(\hat{})]^{1}\hat{} = T \left(1 + \frac{\hat{}^2_m}{\hat{}^2_m}\right)^{1} \hat{}^0\hat{}^{1}\hat{} \qquad {}^2_N$$

where ^ is a consistent estimator of .

F test statistic: Gibbons, Ross and Shanken (1989, GRS) suggest to use an exact distribution by using the F-test (assuming that ε_t's are normally distributed):

$$J_1 = \frac{T-N-1}{N}\left[1+\frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]^{-1}\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} \sim F_{(N;T-N-1)}$$

GRS show that

$$J_1 = \frac{T-N-1}{N}\left(\frac{\left(\frac{\hat{\mu}_q}{\hat{\sigma}_q}\right)^2-\left(\frac{\hat{\mu}_m}{\hat{\sigma}_m}\right)^2}{1+\left(\frac{\hat{\mu}_m}{\hat{\sigma}_m}\right)^2}\right)$$

where q is ex post tangency portfolio, and $\frac{\hat{\mu}_q}{\hat{\sigma}_q}$ the Sharpe ratio. If H_0 holds, market portfolio is the tangency portfolio and J_1 is zero.

Likelihood Ratio test statistic:

$$J_2 = 2\left(\hat{\mathcal{L}} - \tilde{\mathcal{L}}\right) = T\left[\ln|\tilde{\Sigma}| - \ln|\hat{\Sigma}|\right] \sim \chi^2_N$$

where $\hat{\mathcal{L}}$ is the unrestricted log-likelihood and $\tilde{\mathcal{L}}$ is the restricted log-likelihood, i.e. as obtained from the model under the null:

$$X_t = \beta X_{mt} + \varepsilon_t$$

and $\tilde{\Sigma}$ is the restricted estimator:

$$\tilde{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}\left(X_t - \tilde{\beta}X_{mt}\right)\left(X_t - \tilde{\beta}X_{mt}\right)'$$

Notice that J_1 is a monotonic transformation of J_2.
We can use a small sample adjustment to J_2:

$$J_3 = \frac{T - (N/2) - 2}{T} J_2 \sim \chi^2_N$$

J_3 should have better finite-sample properties.